6.B: the ***PhilosophyIdeas*** overview of

Foundations of Mathematics

Mathematicians seek foundations for the subject because they wish to increase confidence in its results, and to unite its diverse branches. Philosophers study the topic because they are interested in the essential nature of mathematics, and its relations to the rest of reality. It is an open question whether mathematics needs foundations; many sports, for example, were played successfully before precise rules were formulated. The quest for foundations, though, has revealed huge areas of rational thought that were hidden from us. The two main rivals for being 'the' foundation of mathematics are the objects it deals with, and the principles used in the dealing. Should we focus on numbers, collections, concepts, shapes and functions, or on logical truths, rules and a priori insights? The ultimate dream was to have a few self-evident principles and concepts, and then prove every part of the subject, but it turned out that the truths of arithmetic will always outrun our ability to prove them. Nowadays the dream is of complete explanations.

If a foundation of a branch of mathematics clearly and satisfyingly explains all of its features, then the ancient **axioms of geometry** were a first attempt. Most of those axioms were interdependent, but the one axiom that wasn't (for parallels) proved flexible; a geometry could have pairs of parallel lines, or none, or several of them. Either there were many geometries, or the parallel axiom was not foundational. It was subsequently shown that some assumptions (about congruence) were not in the ancient axioms, but this was only achieved after geometry was expressed algebraically. Recent axiomatizations present geometry as points, functions and real numbers, though critics say spatial concepts and diagrams are essential. The search for geometrical foundations seems to be inconclusive.

A major development in the quest for foundations was the arrival of a set of simple **axioms for arithmetic**. These say 1) zero is a number, 2) numbers have successors, 3) each number has a unique successor, 4) zero isn't a successor, 5) properties of zero which carry over to successors are properties of all numbers (the 'induction axiom'). These axioms give a starting point, no repetitions or loops, an unending sequence, and no 'stray' numbers. They are accepted as standard because formal models of the axioms all map onto one another (they are 'isomorphic' to one another, so the whole system is 'categorical'), so they completely describe the structure of the numbers. This may be the nearest we can get to the foundations of numbers, but critics are uneasy about having isomorphic models instead of just one model, and (more importantly) are not happy with the use of 'zero', 'number' and 'successor' as unexplained terms. We seem to be offered the structure of ordered numbers, but not their essential nature.

The questions 'how many runners were there?', and 'who came tenth?' point to two concepts of numbers (the **cardinals**, and the **ordinals**), and we can enquire which of these two types of number is more foundational to arithmetic. The arithmetic axioms give priority to the ordinals (because 'successor' is a key concept) and they define an order sequence, where each number is distinguished only by its place in the sequence. **Counting**, which seems closely linked to the fundamentals of arithmetic, would be impossible for us without a standard order to the numbers. The axioms give a simple procedure to arriving at a particular number, whereas pure cardinals stand in isolation, and we might even know some cardinals but not others. Addition might also give priority to ordinals, though angles can also be added together. Defenders of the priority of cardinals point to the very basic principle that two collections are the same size if their members are in **one-to-one** correspondence. No counting or successor relations are involved, and being equinumerous (or of the same 'cardinality') is directly perceived. To arrange objects in height order you must know their heights first, and if numbers concern collections you may need their cardinality before you can order them. The discussion is again inconclusive, but involves what is *psychologically* fundamental about numbers.

A problem for all attempts to give axioms or basic principles for numbers is whether only numbers will fit the account. For example, the axioms above use the word 'number', but actually only define an endless progression of some sort. Similarly, the one-to-one matching relation doesn't reveal why what the groups have in common is their number (just as if we match all the string quartets, they share 'musicality' as much as 'fourness'). The most popular attempt at saying what numbers actually are is to identify them with **sets**. There are many sets containing four items, and they cannot all be the number 'four', but the set of all of those sets seems to embrace every instance of fourness, and so it might constitute the number four. Alternatively, we might say that 'four' is the concept of that family of sets, rather than the sets themselves.

Using the tools of set theory, there have been two notable attempts to define what each number is. If we identify the empty set ∅ with 'zero', we can identify 'one' with its singleton set {∅}, and 'two' with the singleton set of 'one' {{∅}}, and so on. The second account makes 'one' a subset of 'two', so that 'two' is {∅,{∅}}, and 'two' a subset of 'three' {∅,{∅},{∅,{∅}}}. Both versions offer a system for producing a successor, but in the second version 'two' is said to be a part of 'three', which is not so in the first version, so we can consult our intuitions as to which of those seems right. Mathematicians prefer the second version, because it handles infinite numbers better. If one version were accepted, we might claim that we have revealed the foundation of number, but critics say that all sets do is model numbers, rather than say what they are. A famous criticism is that both of these models are equally good, so clearly neither of them can actually be correct, and numbers are more about structural systems than about groups of entities. The structuralist view says that structures (rather than objects) are foundational.

The **structuralist** approach sees foundations in patterns and relations, rather than in the nature of certain abstract objects. There is a division, however, over what is foundational to a structure. One view is that the patterns originate outside of mathematics (perhaps in the physical world). Another view says that patterns are of abstract objects within mathematics, since there are only structures if there are objects to compose the structure. The final view says that structure and relations are everything, though there is still disagreement over the ontology of the structures (as eternally existing, or as conventional or hypothetical). **Category Theory** is an attempt to dig even deeper for foundations, offering nothing but a flow of relations, so generalised that they can defines sets and numbers, and reveal the extreme abstraction of whatever foundations mathematics may ultimately possess.